Q8 / 80

Find $\operatorname{div}(\vec{F}+\vec{g})$ if

$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{g} = x\sin x\,\vec{i} + y\cos y\,\vec{j} + z\sec z\,\vec{k}$.

Sol:- $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{f_1} = x \qquad f_2 = y \qquad f_3 = z$

$\dfrac{\partial f_1}{\partial x} = 1 \qquad \dfrac{\partial f_2}{\partial y} = 1 \qquad \dfrac{\partial f_3}{\partial z} = 1$

$\operatorname{div}\vec{F} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_3}{\partial y} + \dfrac{\partial f_3}{\partial z}$

$\operatorname{div}\vec{F} = 1+1+1$

$\boxed{\operatorname{div}\vec{F} = 3}$

$\vec{g} = x\sin x\,\vec{i} + y\cos y\,\vec{j} + z\sec z\,\vec{k}$

$g_1 = x\sin x$

$\dfrac{\partial g_1}{\partial x} = x\cos x + \sin x \cdot 1$

$g_2 = y\cos y$

$\dfrac{\partial g_2}{\partial y} = y(-\sin y) + \cos y \cdot 1$

$g_3 = z\sec z$

$\dfrac{\partial g_3}{\partial z} = z\cdot\sec z\tan z + \sec z \cdot 1$

$\operatorname{div}\vec{g} = \dfrac{\partial g_1}{\partial x} + \dfrac{\partial g_2}{\partial y} + \dfrac{\partial g_3}{\partial z}$

$\operatorname{div}\vec{g} = x\cos x + \sin x + (-y\sin y + \cos y) + (z\sec z\tan z + \sec z)$

$\operatorname{div}(\vec{F}+\vec{g}) = \operatorname{div}\vec{F} + \operatorname{div}\vec{g}$

$\operatorname{div}(\vec{F}+\vec{g}) = 3 + x\cos x + \sin x - y\sin y + \cos y + z\sec z\tan z + \sec z$

Ans.

LAPLACE OPERATOR

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$\nabla^2 = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

∇^2 is called the laplace operators.

LAPLACIAN OF A SCALAR FUNCTION f

$\nabla^2 f$ is called laplacian of f

Hence $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

LAPLACE'S EQUATION

The equation $\nabla^2 f = 0$ is known as laplace's operator.

HARMONIC FUNCTION

If $\nabla^2 f = 0$

Then the function f is said to be harmonic.

Q 10 / 81 Find the Laplacian of F ∳

(a) $f(x,y,z) = x^2yz + xy^2z + xyz^2$

Sol:- $f(x,y,z) = x^2yz + xy^2z + xyz^2$

$$\frac{\partial f}{\partial x} = 2xyz + y^2z + yz^2$$

$$\frac{\partial^2 f}{\partial x^2} = 2yz + 0 + 0 = 2yz$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 2yz}$$

$$\frac{\partial f}{\partial y} = x^2z + 2xyz + xz^2$$

$$\frac{\partial^2 f}{\partial y^2} = 0 + 2xz + 0 = 2xz$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = 2xz}$$

$$\frac{\partial f}{\partial z} = x^2y + xy^2 + 2xyz$$

$$\frac{\partial^2 f}{\partial z^2} = 0 + 0 + 2xy$$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = 2xy}$$

Laplacian of F

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = 2yz + 2xz + 2xy$$

$$\boxed{\nabla^2 f = 2(yz + xz + xy)}$$ Ans

(b) $f(x,y,z) = yz\cos x + zx\cos y + xy\cos z$.

Sol:- $\nabla^2 f = ?$

Since we know that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{——— (1)}$$

So, $\frac{\partial f}{\partial x} = yz(-\sin x) + z(1)\cos y + y\cos z$

$$\frac{\partial^2 f}{\partial x^2} = -yz\cos x + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = -yz\cos x}$$

$$\frac{\partial f}{\partial y} = (1)\,z\cos x + zx(-\sin y) + x(1)\cos z$$

$$\frac{\partial^2 f}{\partial y^2} = 0 - zx\cos y + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial y^2} = -zx\cos y}$$

$$\frac{\partial f}{\partial z} = y(1)\cos x + (1)\,x\cos y + xy(-\sin z)$$

$$\frac{\partial^2 f}{\partial z^2} = 0 + 0 - xy\cos z \Rightarrow \boxed{\frac{\partial^2 f}{\partial z^2} = -xy\cos z}$$

Putting all these three values in equation (1)

$$\nabla^2 f = -yz\cos x - zx\cos y - xy\cos z$$

$$\boxed{\nabla^2 f = -(yz\cos x + zx\cos y + xy\cos z)}$$

Ans

(Q11/81) If $f = a_1 x^2 + a_2 y^2 + a_3 z^2 + b_1 yz + b_2 zx + b_3 xy$.
where a_1, a_2, a_3 and b_1, b_2, b_3 are all constants.
Show that the Laplacian of f is zero (May be asked as constant)

Sol:- $f = a_1 x^2 + a_2 y^2 + a_3 z^2 + b_1 yz + b_2 zx + b_3 xy$

$$\frac{\partial f}{\partial x} = 2a_1 x + 0 + 0 + 0 + b_2 z + b_3 y$$

$$\frac{\partial^2 f}{\partial x^2} = 2a_1 + 0 + 0 \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = 2a_1}$$

$$\frac{\partial f}{\partial y} = 0 + 2a_2 y + 0 + 0 + 0 + b_3 x$$

$$\frac{\partial f^2}{\partial y^2} = 2a_2 + 0 \Rightarrow \boxed{\frac{\partial f^2}{\partial y^2} = 2a_2}$$

$$\frac{\partial f}{\partial z} = 0 + 0 + 2a_3 z + 0 + 0$$

$$\frac{\partial^2 f}{\partial z^2} = 2a_3 + 0 + 0 \Rightarrow \boxed{\frac{\partial f^2}{\partial z^2} = 2a_3}$$

Laplacian of f

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 2a_1 + 2a_2 + 2a_3$$

$$\boxed{\nabla^2 f = 2(a_1 + a_2 + a_3)}$$ Ans

GRADIENT OF A SCALAR FUNCTION

$\vec{\nabla} f$ is called gradient of the scalar function f.

$$\text{grad } f = \vec{\nabla} f \quad \text{where} \quad \vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

v.v. imp

Q12/81 Prove that $\text{div}(f\vec{\nabla} g) = f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g$

Sol:- L.H.S

$$\text{div}(f\vec{\nabla} g) = \text{div}\left[f\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]g\right]$$

$$= \text{div}\left[f\left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]\right]$$

$$= \text{div}\left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$\boxed{\text{div}(f\vec{\nabla} g) = f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial f}} \quad (1)$$

R.H.S =

$$f(\nabla^2 g) + \vec{\nabla} f \cdot \vec{\nabla} g = f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]$$

$$f(\nabla^2 g) + \vec{\nabla} f \cdot \vec{\nabla} g = f\frac{\partial^2 g}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \quad (2)$$

$$(1) = (2)$$

So $\boxed{\text{div}(f\vec{\nabla} g) = f\vec{\nabla} g + \vec{\nabla} f \cdot \vec{\nabla} g}$ Proved

Q 13/81

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{a}$ is a constant prove

(i) $\operatorname{div}(\vec{a} \times \vec{r}) = 0$

Sol:- $\operatorname{div}(\vec{a} \times \vec{r}) = \vec{\nabla} \cdot (\vec{a} \times \vec{r})$

$= \vec{\nabla} \cdot \vec{a} \times \vec{r}$ (scalar triple product)

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

, where $\vec{a} = [a_1, a_2, a_3]$

Expanding by R_1

$$= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$= 0 - 0 + 0 - 0 + 0 - 0$

$\boxed{\operatorname{div}(\vec{a} \times \vec{r}) = 0}$ proved

V.imp

(ii) $\operatorname{div}[\vec{a} \times (\vec{r} \times \vec{a})] = 2a^2$

Sol:- $\operatorname{div}[\vec{a} \times (\vec{r} \times \vec{a})] = \operatorname{div}[(\vec{a} \cdot \vec{a})\vec{r} - (\vec{a} \cdot \vec{r})\vec{a}]$ (vector triple product)

$= \operatorname{div}[a^2 \cdot \vec{r} - (\vec{a} \cdot \vec{r})\vec{a}]$

$= \operatorname{div} a^2 \cdot \vec{r} - \operatorname{div}(\vec{a} \cdot \vec{r})\vec{a}$ ①

$= a^2 \operatorname{div}\vec{r} - \operatorname{div}(\vec{a} \cdot \vec{r})\vec{a}$

$\operatorname{div}\vec{r} = \vec{\nabla} \cdot \vec{r}$

$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [x, y, z]$

P.T.O

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1$$

$$\text{div}\,\vec{r} = \vec{\nabla}\cdot\vec{r} = 3$$

$$(\vec{a}\cdot\vec{r})\vec{a} = ([a_1, a_2, a_3]\cdot[x, y, z])[a_1, a_2, a_3]$$

$$= (a_1 x + a_2 y + a_3 z)[a_1, a_2, a_3]$$

$$= [(a_1 x + a_2 y + a_3 z)a_1, (a_1 x + a_2 y + a_3 z)a_2, (a_1 x + a_2 y + a_3 z)a_3]$$

$$\text{div}(\vec{a}\cdot\vec{r})\vec{a} = \vec{\nabla}\cdot(\vec{a}\cdot\vec{r})\vec{a}$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot[(a_1 x + a_2 y + a_3 z)a_1, (a_1 x + a_2 y + a_3 z)a_2, (a_1 x + a_2 y + a_3 z)a_3]$$

$$= \frac{\partial}{\partial x}(a_1^2 x + a_1 a_2 y + a_1 a_3 z) + \frac{\partial}{\partial y}(a_1 x + a_2^2 y + a_1 a_3 z)$$

$$+ \frac{\partial}{\partial z}(a_3 a_1 x + a_2 a_3 y + a_3^2 z)$$

$$= a_1^2 + 0 + 0 + 0 + a_2^2 + 0 + 0 + 0 + a_3^2$$

$$= a_1^2 + a_2^2 + a_3^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2$$

$$= |\vec{a}|^2$$

$$\text{div}(\vec{a}\cdot\vec{r})\vec{a} = a^2$$

(1) $\Rightarrow$

$$\text{div}[\vec{a}\times(\vec{r}\times\vec{a})] = a^2(3) - a^2 = 3a^2 - a^2$$

$$\text{div}[\vec{a}\times(\vec{r}\times\vec{a})] = 2a^2 \quad \underline{\text{Ans}}$$

Q15/81 Show $\vec{\nabla}\cdot(\vec{\nabla}f \times \vec{\nabla}g) = 0$

Sol:- $$\vec{\nabla}f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]f$$

$$= \left[\frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f, \frac{\partial}{\partial z}f\right]$$

$$\vec{\nabla}g = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]g$$

$$= \left[\frac{\partial}{\partial x}g, \frac{\partial}{\partial y}g, \frac{\partial}{\partial z}g\right]$$

$$\vec{\nabla}\cdot(\vec{\nabla}f \times \vec{\nabla}g) = \vec{\nabla}\cdot\vec{\nabla}f \times \vec{\nabla}g \quad \text{(scalar triple product)}$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} \quad \text{Expanding by } R_1$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)$$
$$+ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)$$
$$+ \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

$$= \left(\frac{\partial f}{\partial y}\right)\frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial z} - \left(\frac{\partial f}{\partial z}\right)\frac{\partial^2 g}{\partial x \partial y}$$
$$- \frac{\partial^2 f}{\partial x \partial z}\cdot\frac{\partial g}{\partial y} + \left(\frac{\partial f}{\partial z}\right)\frac{\partial^2 g}{\partial y \partial x} + \frac{\partial^2 f}{\partial y \partial z}\cdot\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y \partial z}$$
$$- \frac{\partial^2 f}{\partial y \partial x}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial x}\cdot\frac{\partial^2 g}{\partial z \partial y} + \frac{\partial^2 f}{\partial z \partial x}\cdot\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial z \partial x}$$
$$- \frac{\partial^2 f}{\partial z \partial y}\frac{\partial g}{\partial x}$$

$$\vec{\nabla}\cdot(\vec{\nabla}f \times \vec{\nabla}g) = 0 \quad \underline{\text{proved}}$$